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CUBICAL GRAPHS AND CUBICAL DIMENSIONS

FRANK HARARY
Computing Research Laboratory, New Mexico State University, Las Cruces, NM 88003, U.S.A.

Abstract—A cubical graph G is isomorphic to a subgraph of some hypercube Q_n. The cubical dimension cd(G) is the smallest such n. The induced cubical dimension icd(G) is the minimum n for which G is an induced subgraph of Q_n. The determination for a given cubical graph G of the exact values of cd(G) and icd(G) is very difficult. We discuss these values for some special classes of graphs including trees, unicyclic graphs, "polyomino" animals and polyhexes.

1. INTRODUCTION

A *hypercube* Q_n, or n-cube, is the graph (V_n, E_n) for which the node set V_n is the collection of all binary n-strings, so that $|V_n| = 2^n$. Two nodes $u = u_1 \ldots u_n$ and $v = v_1 \ldots v_n$ are adjacent when they differ in exactly one place. Thus Q_n is n-regular and $|E_n| = n2^{n-1}$ (a number which can be described in jest as the derivative of 2^n with respect to 2). Figure 1 shows the first three Q_n.

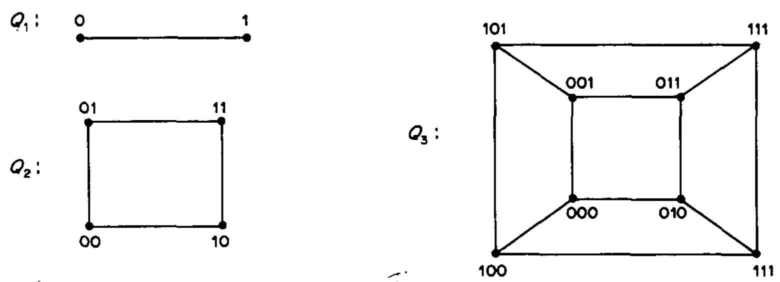

Fig. 1. The three smallest hypercubes.

Alternatively, but equivalently Q_n may be defined recursively using the cartesian product (see [1, p. 22] whose notation and terminology is followed in general):

$$Q_1 = K_2 \quad \text{and} \quad Q_n = Q_{n-1} \times Q_1. \tag{1}$$

Because of the current use of parallel processors having a hypercube architecture, it is of considerable contemporary interest to study graphs which can be so embedded. Following Garey and Graham [2], a *cubical graph* $G = (V, E)$, say with $|V| = p$ and $|E| = q$, is isomorphic to a subgraph of some hypercube Q_n, written $G \subset Q_n$. Then we define the *cubical dimension* written cd(G), of a cubical graph G as the minimum n such that $G \subset Q_n$.

For example, it is trivially seen that all trees and all even cycles are cubical; hence the unicycle graphs with an even cycle are cubical.

The *weight of a node* $u = u_1 \ldots u_n \in V_n$ is Σu_i. Obviously Q_n is bipartite with two color classes determined by the parity of the weight of the nodes. A survey of the theory of hypercubes is given in Harary *et al.* [3].

Thus, every cubical graph is bipartite, but the converse does not hold, as noted by Firsov [4]. The smallest counterexample is provided by the complete bipartite graph $K_{2,3}$. This is seen at once by referring to Fig. 2 where without loss of generality the left node is labeled 000, so the three nodes in its neighborhood have weight 1. Hence, there can be no *other* node adjacent to all three of these.

The cubical dimension of a tree T of order h attains its maximum value when T is a star and its minimum for a path:

$$\text{cd}(K_{1,h-1}) = h - 1 \quad \text{and} \quad \text{cd}(P_h) = \lceil \log_2 h \rceil. \tag{2}$$

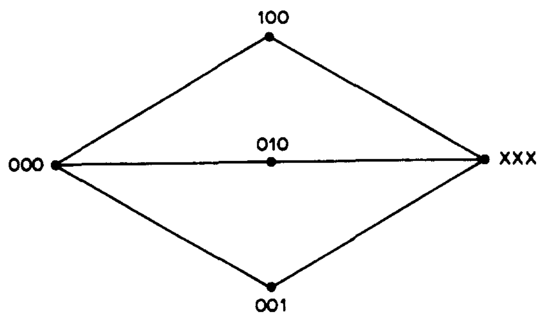

Fig. 2. The smallest bipartite graph which is not cubical.

Similarly, the cubical dimension of an even cycle is obviously given by

$$\mathrm{cd}(C_{2h}) = \lceil \log_2 2h \rceil, \tag{3}$$

as every hypercube Q_n is bipartite pancyclic, i.e., contains cycles of all even lengths from 4 to 2^n.

2. INDUCED CUBICAL DIMENSION

When G is an induced subgraph of H we write $G < H$. The *induced cubical dimension* of a cubical graph G, written $\mathrm{icd}(G)$, is the smallest n such that $G < Q_n$. It is an extremely difficult problem to determine $\mathrm{icd}(G)$ even for the simplest cubical graphs.

Let us define an *n-snake* as a path P_h which satisfies $P_h < Q_n$ but $P_{h+1} < Q_n$, in words, as a longest induced path in Q_n. Similarly, an *n-coil* is a longest induced cycle in Q_n. We denote by s_n (and c_n) the *length* of an n-snake (and an n-coil). These concepts and the determination of the numbers s_n and c_n have been much studied; see Klee [5].

The numbers s_n and c_n are known exactly, Table 1, only through $n = 6$, with the last two values of n found by a computer-aided calculation. When the values for $n = 6$ were found by Davies [6], his conjecture that $2s_n = C_{n+1}$ for all $n \geqslant 3$ was supported. It is most likely that this conjecture will never be proved but it is not impossible that it may be disproved, especially if it is false.

Table 1. The length of snakes and coils

n	s_n	c_n
2	2	4
3	4	6
4	7	8
5	13	14
6	26	26

Table 2. The cubical dimension and induced cubical dimension of small trees T

T	P_2	P_3	P_4	$K_{1,3}$	P_5	$S(112)$	$K_{1,4}$
cd(T)	1	2	3	3	3	3	4
icd(T)	1	2	3	3	3	4	4
T	P_6	$S(113)$	$S(122)$	$C(22)$	$S(1112)$	$K_{1,5}$	
cd(T)	3	3	3	3	4	5	
icd(T)	4	4	4	4	4	5	
T	P_7	$S(114)$	$S(123)$	$S(222)$	$S(1113)$	$S(1122)$	
cd(T)	3	3	3	3	4	4	
icd(T)	4	4	4	4	5	5	
T	$C(14)$	$K_{1,6}$	$C(202)$	$C(112)$	$C(23)$		
cd(T)	5	6	4	4	4		
icd(T)	5	6	4	4	5		

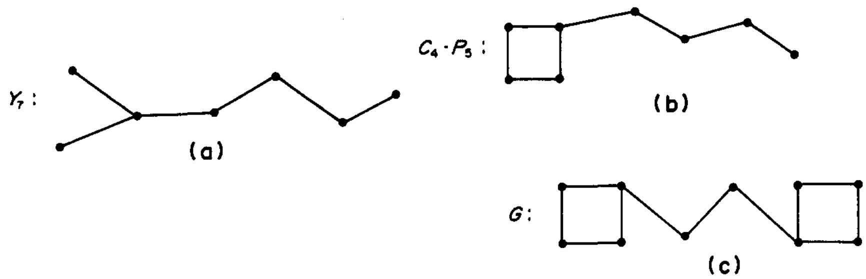

Fig. 3. Three families of cubical graphs.

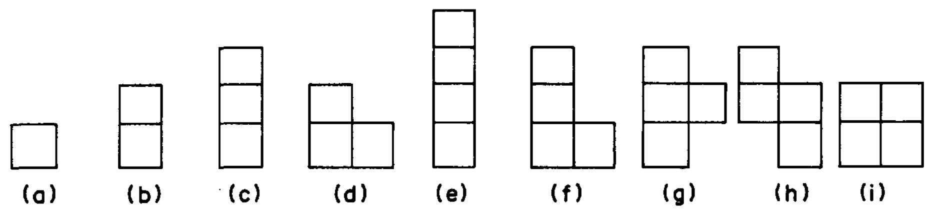

Fig. 4. The smallest animals.

We list in Table 2 the cubical dimension and the induced cubical dimension of all trees of order at most 7. In this listing, as usual, P_h is the path of order h and $K_{1,h}$ is the star with h endnodes. Also, $C(a_1\, a_2 \ldots a_h)$ is the caterpillar with code $(a_1 \ldots a_h)$, i.e., having a_i endnodes adjacent with node v_i of its spine $v_1 \ldots v_h$. Finally, $S(a_1 \ldots a_h)$ is the starlike tree whose paths from the unique node of degree greater than 2 have lengths a_i.

As every path is a caterpillar (with code 100 ... 01), there is no realistic hope for determining icd(T) for an arbitrary caterpillar in terms of its code (or otherwise). However, it may be feasible to derive cd(T) exactly for both caterpillars and starlike trees.

Other families of cubical graphs G suggest themselves for the calculation of the cubical dimension cd(G). These include, as shown in Fig. 3, Y_h, the elongated Y tree of order h, a quadrilateral (or a longer even cycle) with a "tail" P_h attached, and two squares joined by a path.

When the path in Fig. 3(c) is just an edge, then cd(G) = 3 but icd(G) = 5.

3. ANIMALS AND POLYHEXES

The smallest square cell *animals*, also called polynomials by Golomb [7], are shown in Fig. 4 for at most four cells. They served as the goals for various games in a column by Gardner [8].

In Table 3, for each of the nine animals A of Fig. 4, regarded as a plane graph in which every interior face is a square, the dimensions cd(A) and icd(A) are listed.

The *polyhexes* are the corresponding plane graphs in which every interior face is a hexagon. They are of considerable interest in organic chemistry as they have the structure of compounds with multiple benzene rings.

An enumeration problem for polyhexes was solved in [9]. Figure 5 shows the polyhexes P with at most three hexagons. Their cd and icd values are listed in Table 4.

Table 3. The cubical dimensions of the smallest animals A

A	a	b	c	d	e	f	g	h	i
cd(A)	2	3	3	4	5	4	4	4	4
icd(A)	2	3	5	4	5	5	5	5	4

Table 4. The cubical dimensions of small polyhexes

P	a	b	c	d	e
cd(P)	3	4	4	5	5
icd(P)	3	5	6	6	6

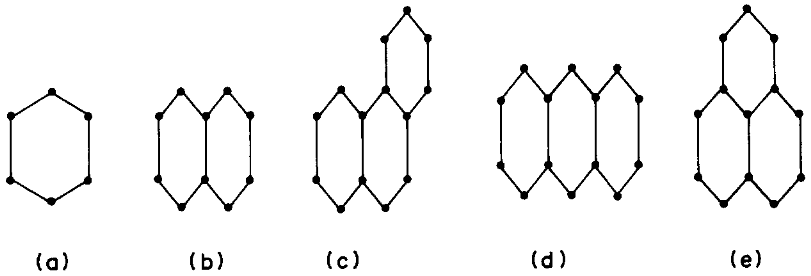

Fig. 5. The smallest polyhexes.

4. FURTHER RESULTS AND OPEN QUESTIONS

The topological cubical dimension, tcd(G) was defined in [10] for an arbitrary graph G (not necessarily cubical) as the minimum n such that for some subdivision H of G, $H \subset Q_n$. It was already noted by Hartman [11], that tcd $(K_p) = p - 1$. Thus, every graph G of order p satisfies tcd$(G) \leqslant p - 1$. Obviously the minimum size of a graph of order p with tcd $= p - 1$ is $p - 1$, as realized by a star, and every graph of order p with maximum degree $\Delta = p - 1$ also has tcd $= p - 1$.

That the cubical dimension is additive for the cartesian product of two graphs is well known and easily proved.

$$\mathrm{cd}(G_1 \times G_2) = \mathrm{cd}(G_1) + \mathrm{cd}(G_2). \tag{4}$$

It follows at once from conditions (3) and (4) that the mesh (or grid) $G_{m,n} = P_m \times P_n$ satisfies

$$\mathrm{cd}(G_{m,n}) = \lceil \log_2 m \rceil + \lceil \log_2 n \rceil, \tag{5}$$

which extends to the product of any number of paths. As a dimensional check, we mention that condition (4) implies cd$(Q_n) = n$.

A *full binary tree of height n* written T_n, can be defined as the rooted tree such that:

(1) the root is at the top of the figure;
(2) the slope of each edge is either $+1$ or -1;
(3) the root has degree 2; the other non-endnodes degree 3;
(4) the distance between the root and every endnode is n.

Hence T_n has 2^n endnodes. The cubical dimension of T_n for $n \geqslant 2$ was derived by Havel and Liebl [12]:

$$\mathrm{cd}(T_n) = n + 2. \tag{6}$$

We conclude with some unsolved problems.

Problem 1

For which cubical graphs G, and in particular for which trees, is cd(G) = icd(G)?

Problem 2

It is easily seen that for $h \geqslant 2$, the cubical dimension of the matching graph is

$$\mathrm{cd}(hK_2) = 2 + \lceil \log_2 h \rceil. \tag{7}$$

What can be said about the cubical dimensions of other linear forests? Of other sets of node-disjoint subcubes?

Problem 3

What is the minimum value of cd(G) among all graphs of order p and size q? What is the maximum value? What are the corresponding extremal values of icd(G)?

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